

[MATHS STUDIES INTERNAL INVESTIGATION]

Investigation of the Golden Ratio in Grade 12

College Students

0

Introduction

The Golden Ratio

The golden ratio is also known as phi (ϕ) and refers to the ratio of 1:1.618, which is considered aesthetically pleasing to the human eye. The diagram below demonstrates this ratio; $\phi-1:\phi$ is a ratio of 1:1.618.

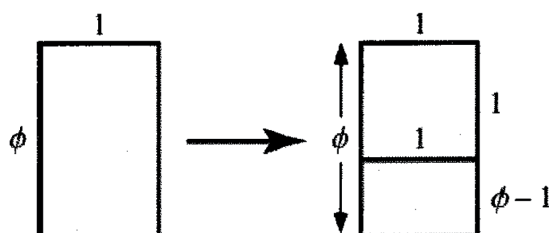


Fig. 1 – representation of the golden ratio

The golden ratio can be applied to many situations and in this investigation it will be explored in relation to facial features. The ratio of 1:1.618 is considered to be the most aesthetically pleasing facial ratio and therefore according to the research on this subject, people who are ‘beautiful’ will have a ratio that is close to 1:1.618.

Aim

The aim of this investigation is to establish the frequency of the golden ratio among grade 12 students, to compare the prominence of the facial golden ratio in males versus females and to determine the independence of the facial golden ratio to gender.

Investigational Procedure

This investigation will be conducted by calculating the facial golden ratios of a sample of 30 males and 30 females, all in grade 12 College, 2008. The investigation will be divided into two sections; Section 1 and Section 2. In Section 1 the data will be examined collectively as a co-gender group of grade 12s and then in Section 2 the results will be separated according to gender and compared with the other sex. The data will be analysed by constructing a frequency table for the entire sample group and then by gender. In these same categories the results will be represented pictorially through frequency polygons and box plots and then the independence will be calculated using the X^2 test. These results will be analysed for trends and summarized in the conclusion.

The hypothesis is that females will show a higher prominence of the facial golden ratio and that it will be dependent on gender.

Data Collection

To collect these data, photographs were taken of the students and then the digital photographs were enlarged on a computer screen to 23cm x 16cm and measurements were taken from these images. The 60 sample students, 30 male and 30 female, were randomly chosen from a list of Grade 12 students. The year group had 249 students with approximately 134 males and 115 females meaning that the sample group represents 24.1% of the year group, 22.4% of males and 26.1% of females.

Care was taken to angle the camera in a horizontal plane to the student's face and the students were asked to look at the camera directly, not to smile and to pull their hair away from their face.

The results were calculated using the following process, which gives the data from a female student, S.D. Due to the precise values being calculated the data were recorded to 4 significant figures.

Name:

Female

- a = Top-of-head to chin = 15.9 cm
- b = Top-of-head to pupil = 7.2 cm
- c = Pupil to nosetip = 2.9 cm
- d = Pupil to lip = 4.5 cm
- e = Width of nose = 3.1 cm
- f = Outside distance between eyes = 7.5 cm
- g = Width of head = 9.5 cm
- h = Hairline to pupil = 5.9 cm
- i = Nosetip to chin = 6 cm
- j = Lips to chin = 2.7 cm
- k = Length of lips = 4 cm
- l = Nosetip to lips = 2 cm

Ratios:

- $a/g = 1.673$ cm
- $b/d = 1.6$ cm
- $i/j = 2.222$ cm
- $i/c = 2.068$ cm

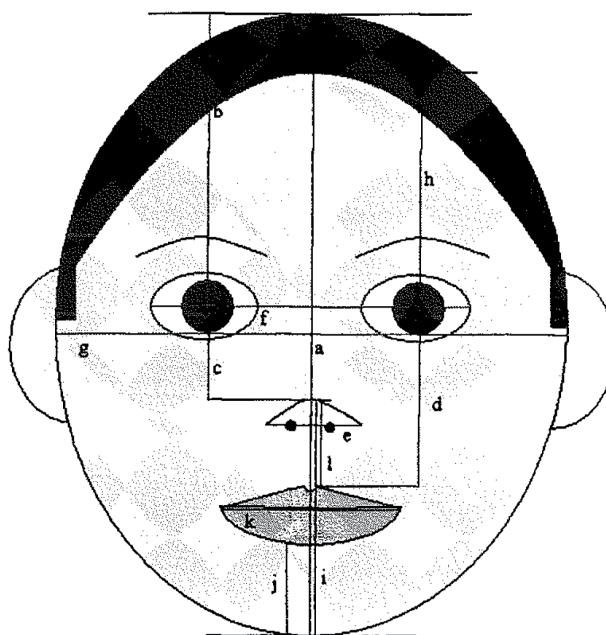


fig. 2 – calculations of faces

$$e/l = 1.55 \text{ cm}$$

$$f/h = 1.271 \text{ cm}$$

$$k/e = 1.290 \text{ cm}$$

Average Ratio: 1:1.667 cm

Results

The calculations for the average ratios (as seen above) are shown in Appendix 1 and the summary of their calculated ratios can be seen in the table below.

Table 1 – Average Ratios of Sample Year 12 Students

Female Students from Sample	Average Ratio (1:)	Male Students from Sample	Average Ratio (1:)
X. B.	1.692	D. T.	1.809
D. R.	1.783	J. M.	1.803
F. S.	1.7	L. S.	1.778
S. P.	1.863	N. A.	1.868
N. O.	1.743	S. C.	1.754
A. T.	1.755	N. W.	1.673
J. H.	1.636	F. W.	1.669
A. O.	1.682	N. W.	1.818
K. B.	1.69	K. R.	1.742
H. L.	1.826	D. D.	1.873
R. W.	1.692	C. S.	1.94
L. Z.	1.75	J. R.	1.717
S. M.	1.669	C. R.	1.765
K. G.	1.7	M. N.	1.813
K. W.	1.682	L. G.	1.661
A. K.	1.642	S. Y.	1.762
S. G.	1.748	D. C.	1.605
J. B.	1.395	D. Y.	1.753
J. R.	1.637	H. E.	1.706
A. P.	1.516	B. H.	1.643
G. M.	1.741	L. S.	1.681
K. P.	1.718	S. M.	1.775
J. W.	1.852	C. F.	1.677
H. W.	1.717	M. S.	1.724
J. R.	1.703	J. L.	1.741
K. V.	1.962	P. C.	1.836
K. C.	1.804	T. S.	1.753
O. A.	1.756	S. M.	1.779
S. D.	1.667	A. L.	1.885
A. F.	1.94	C. G.	1.875

Section 1

To make the data easier to analyse, they were sorted into 13 groups, ranging from 1:1.35 – 1:2.0 and rising in increments of 0.05. The chosen groupings were small due to the precise value being searched for. The data were sorted into groups by the how frequently it occurred. As the golden ratio is 1:1.618 the data in the near vicinity of this was named as having the golden ratio. Any facial ratio that falls between 1:1.55 and 1:1.65 was considered acceptable. This leeway of 0.1 is making allowance for errors and follows the theory that those close to the golden ratio must also be ‘beautiful’ and therefore are adequate for this investigation.

From this the cumulative frequency, relative frequency and cumulative relative frequency were calculated.

Jones, J. (2008) defines the following terms,

Cumulative Frequency

The number of values less than the upper class boundary for the current class. This is a running total of the frequencies.

Relative Frequency

The frequency divided by the total frequency. This gives the percent of values falling in that class.

Cumulative Relative Frequency (Relative Cumulative Frequency)

The running total of the relative frequencies or the cumulative frequency divided by the total frequency. This gives the percent of the values which are less than the upper class boundary.

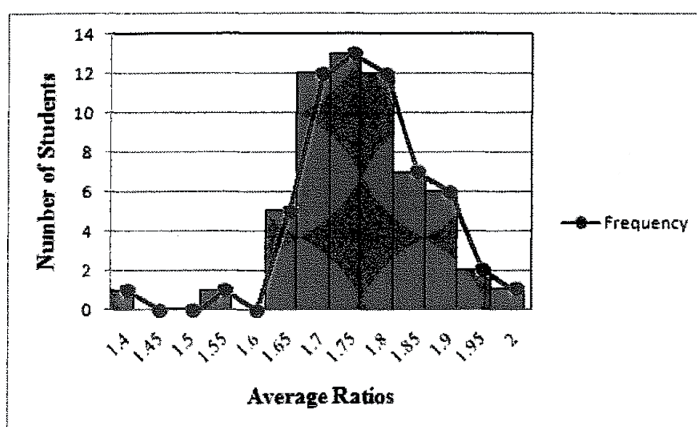
Table 2 - Frequency Table for the Average Facial Ratios of Sample Year 12 Students

Average Ratios	Tally	Frequency	Cumulative Frequency	Relative Frequency	Cumulative Relative Frequency
$1.35 \leq AR < 1.4$	I	1	1	0.017	0.017
$1.4 \leq AR < 1.45$	0	0	0	0	0
$1.45 \leq AR < 1.5$	0	0	0	0	0
$1.5 \leq AR < 1.55$	I	1	2	0.017	0.034
$1.6 \leq AR < 1.65$	HHH	5	7	0.083	0.117
$1.65 \leq AR < 1.7$	HHH HHH II	12	19	0.2	0.317
$1.7 \leq AR < 1.75$	HHH HHH III	13	32	0.216	0.533
$1.75 \leq AR < 1.8$	HHH HHH II	12	44	0.2	0.733
$1.8 \leq AR < 1.85$	HHH II	7	51	0.116	0.849
$1.85 \leq AR < 1.9$	HHH I	6	57	0.1	0.949
$1.9 \leq AR < 1.95$	II	2	59	0.034	0.983
$1.95 \leq AR < 2.0$	I	1	60	0.017	1
	Σ	60			

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The results show that the most common facial ratios of the sample year 12 group occurs between 1:1.7 and 1.75, as it has the highest frequency of 13/60 and therefore the highest relative frequency of 0.216 or 21%. However ratios between 1:1.65 and 1:1.7 and 1:1.75 and 1:1.8 were also common, both having a frequency of 12, and a relative frequency of 0.2 or 20%. Only 8.3% of the students have a facial ratio between 1:1.6 – 1:1.65 which is the group containing the golden ratio. Below is a visual representation of these findings.

Fig. 3 - Frequency Polygon of the Average Facial Ratios of Sample Year 12 Students



This graph has a negative skew when the data from 1:1.4 – 1:1.5 are included, however if these points are considered outliers the data then has only a slight positive skew, with a relatively even spread. This means that most of the data falls into the modal group of 1:1.7 – 1:1.75 and the rest is centred on this group decreasing as the ratio rises.

To verify these findings, the averages of the data were calculated, along with the standard deviation, the minimum, the maximum, the range and the upper and lower quartiles.

Definition of Terms

The **mean** is a calculated average, obtained by dividing the sum of the data by the quantity of the data.

Eg. The mean of $\{1,2,3,4,5\} = (1 + 2 + 3 + 4 + 5) \div 5 = 3$

The **median** is the central number of the data when placed in numerical order.

Eg. The median of $\{1,2,3,4,5\} = 3$

The **mode** is the number which occurs the most frequently.

Eg. The mode of {1,2,3,4,5,5} = 5

The **standard deviation** is “A measure of the dispersion of the frequency distribution” (Thomson Financial, 2004). The formula for standard deviation can be seen below.

$$\sigma = \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}}$$

The **range** is calculated by taking the minimum number away from the maximum number.

The **lower quartile** or Q_1 is the 25th percentile and can be found using the following formula;

$$Q_1 = \frac{n+1}{4}$$

The **upper quartile** or Q_3 represents the 50th percentile and can be found at $3 \times Q_1$.

The **interquartile range** is found by taking Q_3 from Q_1 .

Outliers are values outside the limits of $Q_1 - 1.5 \times \text{IQR}$ and $Q_3 + 1.5 \times \text{IQR}$.

Table 3 - Calculations for the Average Facial Ratios of Sample Year 12 Students

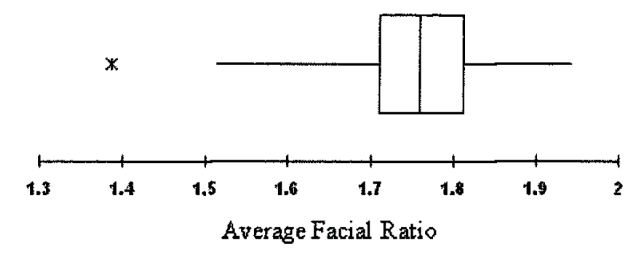
Calculations	Year 12 Students
Mean	1.742
Median	1.7425
Mode(s)	1.669, 1.717, 1.741, 1.775
Standard Deviation	0.098
Minimum	1.395
Maximum	1.962
Range	0.567
Q_1	1.682
Q_3	1.80325
Interquartile Range	0.12125
Outliers	1.395

From these calculations we can see that the mean facial ratio for year 12 students is 1:1.742, this is very similar to the median ratio of 1.7425 and this indicates that the golden ratio is not the

most prominent ratio. The modal averages are all higher than the golden ratio, and this supports the trend seen in this investigation that the majority of the students have a higher facial ratio than the golden ratio. The standard deviation is relatively small indicating that the data have a small spread, which is consistent to the small range of 1:0.567. From this it can be hypothesised that human faces are similar in ratio, even if the majority do not have the golden ratio. However there was one outlier, which fell below the value of 1.5. The outlier to these data set was 1.395.

The quartiles can be used to create a box and whisker plot, displaying the spread of the data, where the median is the 50th percentile, or Q_2 .

Fig. 4 - Box and Whisker Plot for the Average Facial Ratios of Sample Year 12 Students



Although it can be seen that the data have a negative skew, the skew is slight and the data are almost symmetrical, indicating an even spread among the students. To further investigate the spread of students, the data were re-examined according to gender and are outlined in Section 2 below. The interquartile range is 0.12125, from 1.682 to 1.80325. This shows that the data has a small spread and that most facial values are similar, supporting the standard deviation.

Section 2

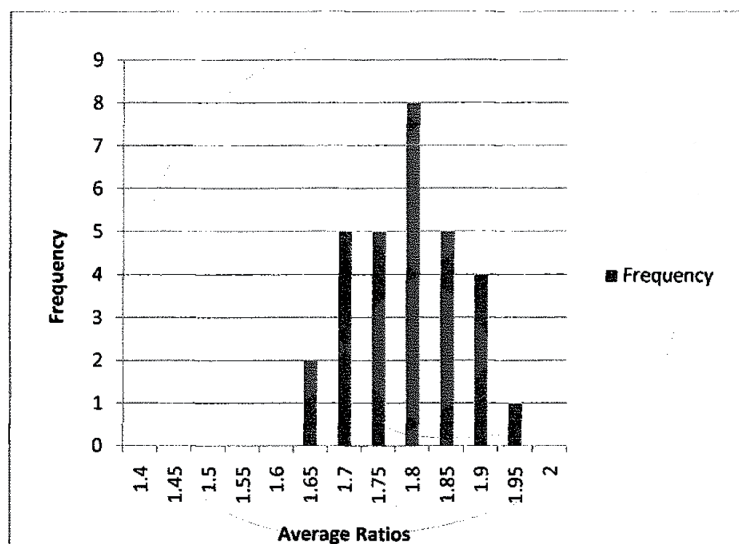
The next analysis involves looking at the separation between male and female students. As with part one, the data were separated into 13 groups according to frequency and from this the cumulative frequency, relative frequency and the cumulative relative frequency were calculated.

Table 4 - Frequency Table for the Average Facial Ratios of Female Students

Average Ratios	Tally	Frequency	Cumulative Frequency	Relative Frequency	Cumulative Relative Frequency
$1.35 \leq AR < 1.4$	I	1	1	0.033	0.033
$1.4 \leq AR < 1.45$	0	0	0	0	0
$1.45 \leq AR < 1.5$	0	0	0	0	0
$1.5 \leq AR < 1.55$	I	1	2	0.033	0.067
$1.6 \leq AR < 1.65$	III	3	3	0.1	0.167
$1.65 \leq AR < 1.7$	HHH II	7	12	0.233	0.4
$1.7 \leq AR < 1.75$	HHH III	8	20	0.267	0.67
$1.75 \leq AR < 1.8$	IIII	4	24	0.133	0.8
$1.8 \leq AR < 1.85$	II	2	26	0.067	0.867
$1.85 \leq AR < 1.9$	II	2	28	0.067	0.933
$1.9 \leq AR < 1.95$	I	1	29	0.033	0.967
$1.95 \leq AR < 2.0$	I	1	30	0.033	1
	Σ	30			

From these results we can see how the female students influenced the whole as the majority group is 1:1.7 – 1:1.75, with a frequency of 20/30 and a relative frequency of 0.267 or 26.7%. However the ratios generally fall into the lower categories and the frequency of ratios between 1:1.75 and 1:1.8 has decreased, which can be seen in the following graph.

Fig. 5 - Frequency Polygon of the Average Facial Ratios of Female Students



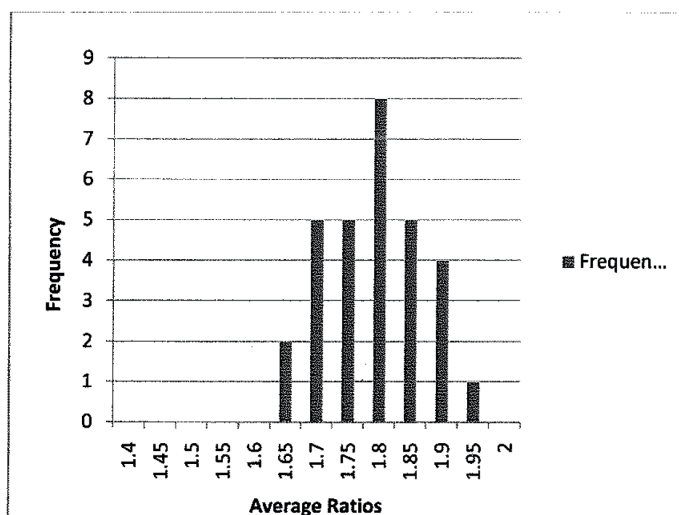
This graph of the average facial ratios of female students is quite symmetrical but has a slight positive skew. Then the same procedure was repeated for the male students on the next page

Table 5 - Frequency Table for the Average Facial Ratios of Male Students

Average Ratios	Tally	Frequency	Cumulative Frequency	Relative Frequency	Cumulative Relative Frequency
$1.35 \leq AR < 1.4$	0	0	0	0	0
$1.4 \leq AR < 1.45$	0	0	0	0	0
$1.45 \leq AR < 1.5$	0	0	0	0	0
$1.5 \leq AR < 1.55$	0	0	0	0	0
$1.6 \leq AR < 1.65$	II	2	2	0.067	0.067
$1.65 \leq AR < 1.7$	HHH	5	7	0.167	0.233
$1.7 \leq AR < 1.75$	HHH	5	12	0.167	0.4
$1.75 \leq AR < 1.8$	HHH III	8	20	0.267	0.67
$1.8 \leq AR < 1.85$	HHH	5	25	0.167	0.833
$1.85 \leq AR < 1.9$	IIII	4	29	0.133	0.967
$1.9 \leq AR < 1.95$	I	1	30	0.033	1
	Σ	30			

The majority of the male average facial ratios fell between 1:1.75 and 1:1.8. This differs from both the female and the overall results as the modal group of these was 1:1.7 – 1:1.75. The male data contributes to the even spread shown in the overall grade 12 results (fig. 4) as it adds to the frequency of the higher groups of ratios which are not common in the female students

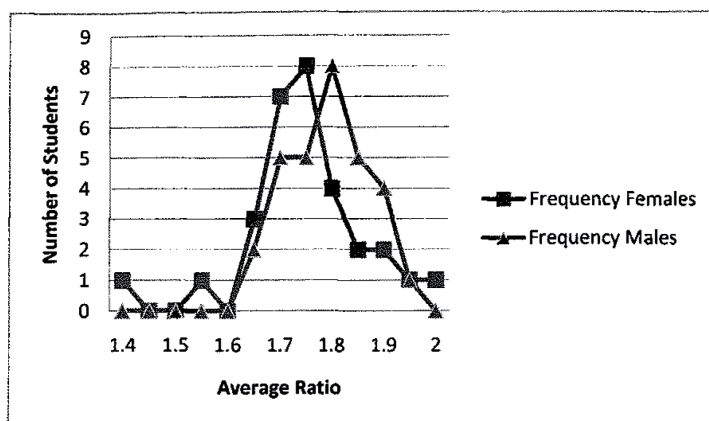
Fig. 6 - Frequency Polygon of the Average Facial Ratios of Male Students



This graph shows the data are more symmetrical than female values.

A line graph was created in order to visually compare the results and can be seen in Fig. 7.

Fig. 7 - Line Graph of the Average Facial Ratios of Male and Female Students



When comparing this graph to the frequency polygon of the entire sample group (fig. 3), we can see how the different genders contributed to the spread of the data. The female sample’s modal class has a smaller ratio than the male sample, but when added together they create a roughly even spread.

The data were placed in a stem and leaf plot. This makes the data easier to sort and provides a visual contrast and comparison of the two groups.

Fig. 9 - Stem and Leaf Plot of the Average Facial Ratios of Male Students vs. Female Students

Male	Female
	1.3 95
	1.4
	1.5 16
81, 77, 73, 69, 61, 43, 05	1.6 36, 37, 42, 67, 69, 82, 82, 92, 92, 9
79, 78, 75, 65, 62, 54, 53, 53, 42, 41, 24, 17, 06	1.7 00, 00, 03, 17, 18, 41, 43, 48, 5, 55, 56, 83
85, 75, 73, 68, 36, 18, 13, 09, 03	1.8 04, 26, 52, 63
04	1.9 62, 94

In this plot, the data were placed into numerical order and this helped to find the mode and also due to the larger size of the groups (increments of 0.1) we can confirm into which categories the majority of the data fell and this plot reveals that 1:1.7 – 1:1.8 is the most frequent ratio in both males and females. It is also apparent that despite having a common modal group, higher facial ratios are more prominent in males, while lower facial ratios are more prominent in females.

As in section 1, the averages, standard deviation, minimum, maximum, range, upper and lower quartile and the interquartile range were calculated.

Table 6 - Calculations for the Average Facial Ratios of Male and Female Students

Calculations	Female Students	Male Students
Mean	1.722	1.763
Median	1.71	1.758
Mode(s)	1.682, 1.692, 1.7	1.753
Standard Deviation	0.11	0.08
Minimum	1.395	1.605
Maximum	1.962	1.94
Range	0.567	0.335
Q ₁	1.682	1.709
Q ₃	1.756	1.812
Interquartile Range	0.074	0.103
Outliers	$1.57 < x < 1.87 = 1.395,$ 1.516, 1.962, 1.994	$1.55 < x < 1.97 =$ no outliers present

The mean facial ratio for females is 1:1.722 and for males 1:1.763 giving a 0.04 difference between the two which, given the small-scale nature of this investigation this is a large difference. The median averages have a difference of 0.48 between the male median (1:1.758) and the female median (1:1.71), which emphasises how the average male facial ratio is higher than the average female ratio. However, none of these ratios are close to the golden ratio of 1:1.618. The standard deviation for females is 0.11 and this is higher than the male standard deviation of 0.08 by 0.03, suggesting that the females are more dispersed than the males. This is supported by the ranges, as the female range is larger than the male range by 0.232, which is a large difference. However, according to the calculations for outliers, the female data must be re-evaluated as 1.395, 1.516, 1.962 and 1.994 must be excluded from the results as they fall into this category.

Table 7 - Calculations for the Average Facial Ratios of Female Students without Outliers

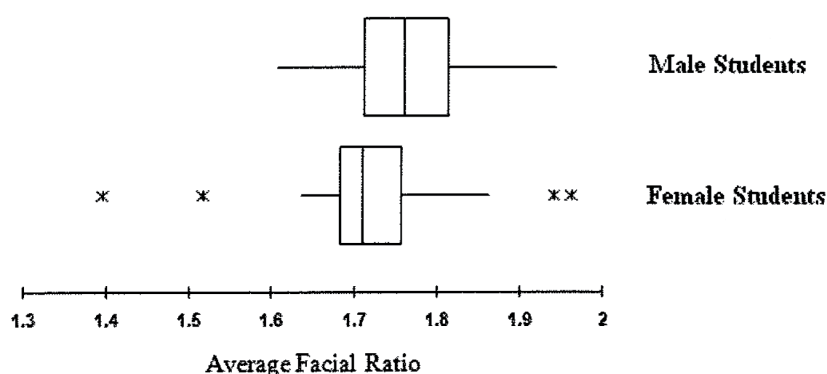
Calculations	Female Students
Mean	1.686
Median	1.71
Mode(s)	1.682, 1.692, 1.7
Standard Deviation	0.062
Minimum	1.636
Maximum	1.863
Range	0.227

With the female outliers excluded, the result is the opposite as before. The females standard deviation is now 0.018 smaller than the male standard deviation and the range is 0.108 smaller

than the male range. The female mean ratio of 1:1.686 is closer to the golden ratio than the male mean facial ratio of 1:1.736. The modal ratios of the female students (1.682, 1.692, 1.7) are also lower than the modal ratio of the male students (1.753) and therefore closer to the golden ratio.

From the quartiles a box and whisker plot can be constructed to compare the findings.

Fig. 10 - Box and Whisker Plot Comparing Average Facial Ratios of Sample Male and Female Students in Year 12



This box plot supports the previous results. The male spread is more even and symmetrical, where the female data has a positive skew, which can be clearly seen on this plot. The female data also have 4 outliers compared to the 0 outliers shown in the male data, suggesting a larger variety in female facial ratios despite the fact that they are classed as outliers. Male students appear to have, on average, a higher facial ratio than females, as the box plot depicts. This is shown by Q_1 of the male data being 1.709, which is nearly the median value of the female data (1.71). The median male facial ratio (1.758) is approximately the same as the female upper quartile (1.756). The interquartile range of the male students (0.103) is larger than the female range of 0.072 indicating that male facial ratios differ more than the female ratios.

The above trends were subjected to statistical analysis using a chi-squared test to determine whether gender and presence of the golden ratio are independent.

Chi-Squared Test

The purpose of a Chi-Squared test is to compare two sets of data and to determine if they are independent of each other. "The idea behind this test is to compare the observed frequencies with the frequencies that would be expected if the null hypothesis of no association / statistical independence were true" (Easton V.J. and McColl J.H. 1997). The null hypothesis is rejected if

$$X^2_{\text{calc}} \text{ is greater than } k. \quad X^2_{\text{calc}} = \sum \frac{(f_o - f_e)^2}{f_e}$$

Step 1

H_0 is presence of the golden ratio and gender are independent

H_1 is presence of the golden ratio and gender are dependent

Step 2

$$Df = (r - 1)(c - 1)$$

$$Df = (2 - 1) \times (2 - 1)$$

$$Df = 1$$

Step 3

Therefore, this data has a 5% significance value

Step 4

We reject H_0 if $X^2_{\text{calc}} > 3.84$

Step 5**Table 8 - Contingency Table**

	Have golden ratio (1:1.5 < x < 1:1.65)	Do not have golden ratio	Sum
Male	2	28	30
Female	3	27	30
Sum	5	55	60

Table 9 - Expected Value Table

	S_1	S_2	Sum
Male	2.5	27.5	30
Female	2.5	27.5	30
Sum	5	55	60

Table 10 - Frequency Calculations

f_o	f_e	$(f_o - f_e)$	$(f_o - f_e)^2 \div f_e$
2	2.5	-0.5	0.1
28	27.5	0.5	0.009
3	2.5	0.5	0.1
27	27.5	-0.5	0.009
Total:			0.218

Step 6

$$X^2_{0.05} = 3.84$$

As $X^2_{\text{calc}} < 3.84$, we accept H_0 in favour of H_1 , i.e. that the *presence of the golden ratio* and *gender* are independent classifications

Step 7

$$P = 0.64$$

As $p > 0.05$ we accept the null hypothesis

Errors and Limitations

As stated in Section 1, the measurements for the average facial ratio were taken from photographs. This would not be as accurate as the more lengthy procedure of calculating every student's measurements directly. However for a secondary student investigation, this was impractical. While it was attempted to take perfectly direct, face on photos, human error of judgement by the investigator and/or the student may have affected the results. The investigator may have taken a photo from a skewed position in relation to the student and the student may have angled their face and not looked directly at the camera. Another problem involving human error was trying to get the students not to smile in the photos and this was surprisingly difficult. All of these would have altered the facial proportions in the photos.

There is a possibility of human error too in the measurement of the photographs. This was reduced by projecting the photos at a large scale (23cm x 16cm), however some error in calculations and measurement must be made allowance for. Also some measurements were difficult to measure due to the limitations of the photographs, such as measuring from the top of the head as this was made obscure by the hair of the students.

The age of the students was not taken into account and faces may change structurally as they age and therefore this is a variable that has not been controlled in this investigation.

Improvements

The main improvement would be to measure the students directly with callipers. Despite being time consuming for the student, it would provide more accurate results, as it would eliminate the incorrect proportions given by a photograph such as the difficulties in measuring from the top of the head, which is hard to determine due to different hairstyles. The sample size of this investigation was approximately 20% and this means that the results are reasonably trustworthy and adequate for the predicting results for grade 12s at College. However, if the investigation applied to the wider community of grade 12s then a larger sample group would be needed. Another improvement would be to only measure those of the same age, as they would most likely be at the same developmental stage. Alternatively an expansion of the investigation could be to separate the age groups and measure accordingly to determine the effect of age on the golden ratio.

C4

E1

Conclusion

It was hypothesised that females would show a higher prominence of the facial golden ratio and that it would be dependent on gender. The former hypothesis was supported while the latter was not.

In summary, the results showed that females did have a higher tendency towards having the golden facial ratio but this trend did not reach statistical significance. Therefore the null hypothesis indicating the presence of the golden ratio and gender are independent was supported. However with improvements in the experimental procedures and a larger sample size it is possible that the trend seen in the results might have reached statistical significance. Such further investigation would reveal whether experimental errors and small sample size limited the ability to support the research hypothesis of gender dependency.

In the absence of a larger survey with better experimental methods, this investigation enables us to predict that the majority of the grade 12 students at _____ College will not have the facial golden ratio, whether male or female, but that the female students will possess a ratio that is lower and therefore closer to the desired 1:1.618.

Bibliography

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Appendix 1

Name:

Male/Female

- a = Top-of-head to chin = 15.9 cm
 b = Top-of-head to pupil = 2.2 cm
 c = Pupil to nosetip = 2.9 cm
 d = Pupil to lip = 4.5 cm
 e = Width of nose = 3.1 cm
 f = Outside distance between eyes = 7.5 cm
 g = Width of head = 9.5 cm
 h = Hairline to pupil = 5.9 cm
 i = Nosetip to chin = 6 cm
 j = Lips to chin = 2.7 cm
 k = Length of lips = 4 cm
 l = Nosetip to lips = 2 cm

Ratios:

- a/g = 1.673 cm
 b/d = 1.6 cm
 i/j = 2.222 cm
 i/c = 2.068 cm
 e/l = 1.55 cm
 f/h = 1.271 cm
 k/e = 1.290 cm

Average Ratio: 1.667 cmTaken from: <http://cuip.net/~dlnarain/golden/activity8.htm> (5/11/07)